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Best Approximation in Finite Dimensional Subspaces of $\mathscr{L}(W, V)$

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We prove Kolmogorov's type characterization of best approximation for given $L \in \mathscr{L}(W, V)$ in finite dimensional subspace $\mathscr{V} \subset \mathscr{L}(W, V)$. This extends the results obtained by Malbrock for the case $W = V = c_0$ and $W = C(T)$, $V = C(S)$.

1. INTRODUCTION

Let X be a normed space over a field K ($K = \mathbb{R}$ or $K = \mathbb{C}$) and let S_{X^*} denote the unit sphere in X^*. For $x \in X$ put

$$E(x) = \{ f \in \text{ext } S_{X^*} : f(x) = \|x\| \} \tag{1.1}$$

(ext W denotes the set of all extremal points of a given set W), and let for $Y \subset X$

$$\mathscr{P}_Y(x) = \{ y \in Y : \|x - y\| = \text{dist}(x, Y) \}. \tag{1.2}$$

If Y is a linear subspace of X then the following Kolmogorov type characterization holds true.

THEOREM 1.1 (see [2]). *Assume X is a normed space, $Y \subset X$ is its linear subspace, and let $x \in X \setminus Y$. Then $y_0 \in P_Y(x)$ if and only if for every $y \in Y$ there exists $f \in E(x - y_0)$ with $\text{re} f(y) \leqslant 0$.*

A similar result can be proved in the case of strong unicity. In order to present it, let us recall that an element $y \in Y$ is called a strongly unique best approximation (briefly, SUBA) for $x \in X$ if and only if there exists $r > 0$ such that for every $y \in Y$,

$$\|x - y\| \geqslant \|x - y_0\| + r \cdot \|y - y_0\|. \tag{1.3}$$

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In [11, Theorem 2.1, p. 855] the following was shown.

THEOREM 1.2. *Let $x \in X \setminus Y$ and let Y be a linear subspace of X. Then $y_0 \in Y$ is a SUBA for x with a constant $r > 0$ if and only if for every $y \in Y$ there exists $f \in E(x - y_0)$ with $\operatorname{re} f(y) \leqslant -r \|y\|$.*

If Y is a finite dimensional subspace of X, then by [10, Theorem 1.1, p. 170] and Theorem 1.2 we get

THEOREM 1.3. *Assume X is a normed space and $Y \subset X$ is a finite-dimensional linear subspace, and let $x \in X \setminus Y$. Then $y \in P_Y(x)$ (resp., y is a SUBA for x in Y) if and only if $0 \in \operatorname{conv} E(x-y)|_Y$ (resp., $0 \in \operatorname{int conv} E(x-y)|_Y$, where $E(x-y)|_Y = \{f|_Y : f \in E(x-y)\}$). (The symbols* conv A *and* int A *denote respectively the smallest convex set containing A and the interior of A with respect to the norm topology.)*

In this note we consider the case when $X = \mathscr{L}(W, V)$ (the space of all linear continuous operators from a normed space W into a normed space V equipped with the operator norm) and $\mathscr{V} \subset X$ is a finite-dimensional subspace. We prove Kolmogorov's type characterization of best approximants (Theorem 2.1) which involves only elements from the sets S_W and ext S_{V^*}. (Note that a similar characterization for the case of compact operators was shown in [5].) We also present a result concerning strong unicity. This extends the results obtained in [7] and [8] for the spaces $W = V = c_0$ and $W = C(S)$, $V = C(T)$. Next we characterize finite-dimensional Chebyshev subspaces in the space $\mathscr{K}(c_0)$ of all compact operators going from c_0 into c_0.

2. GENERAL CASE

Now we formulate the main result of this section.

THEOREM 2.1. *Let W, V be arbitrary normed linear spaces (we consider the real and complex case) and let $\mathscr{V} \subset \mathscr{L}(W, V)$ be an n-dimensional subspace. Assume $L \in \mathscr{L}(W, V) \setminus \mathscr{V}$ and $V_0 \in \mathscr{V}$. Then $V_0 \in P_{\mathscr{V}}(L)$ if and only if for every $\varepsilon > 0$ there exists $m \in N$, $\varphi_1, ..., \varphi_m \in \operatorname{ext} S_{V^*}$, and $w_1, ..., w_m \in S_W$ such that*

$$0 \in \operatorname{conv}\{\varphi_1 \otimes w_1 |_{\mathscr{V}}, ..., \varphi_m \otimes w_m |_{\mathscr{V}}\} \tag{2.1}$$

and

$$\left| \sum_{i=1}^{m} \lambda_i (\varphi_i \otimes w_i)(L - V_0) - \|L - V_0\| \right| \leqslant \varepsilon, \tag{2.2}$$

where $\lambda_i > 0$, $\sum_{i=1}^{m} \lambda_i = 1$. (We set $(\varphi_i \otimes w_i)(L) = \varphi_i(Lw_i)$.)

Proof. Fix $\varepsilon > 0$ and let $\mathscr{L} = [L] \oplus \mathscr{V}$. Since $\mathscr{L}$ is finite dimensional, $S_{\mathscr{L}}$ is a compact set. Hence there exist $C_1, ..., C_m \in S_{\mathscr{L}}$ such that $S_{\mathscr{L}} \subset \bigcup_{i=1}^{m} B_d(C_i, \varepsilon/3)$. (The symbol $B_d(x, r)$ denotes the closed ball with a centre x and a radius r.) Select for each $i \in \{1, ..., m\}$, $\varphi_i \in \operatorname{ext} S_{V^*}$ and $w_i \in S_W$ with

$$| \, \|C_i\| - \varphi_i(C_i w_i)| \leqslant \varepsilon/3. \tag{2.3}$$

Denote $Z_1 = \{\varphi_i \otimes w_i : i = 1, ..., m\}$ and $T = \{\varphi \otimes w : \varphi \in \operatorname{ext} S_{V^*},\ w \in S_W\}$. Note that $T|_{\mathscr{L}}$ is a total set over $\mathscr{L}$. Hence we can choose $Z_2 \subset T|_{\mathscr{L}}$, $Z_2 = \{(\gamma_1 \otimes u_1)|_{\mathscr{L}}, ..., (\gamma_{n+1} \otimes u_{n+1})|_{\mathscr{L}}\}$ which forms a basis of $\mathscr{L}^*$. Put

$$Z = Z_1 \cup Z_2, \tag{2.4}$$

and let $\mathscr{M} = \Gamma Z =$ absolutely convex hull of Z. Since $\mathscr{M}$ is an absolutely convex absorbing set, we can define $\| \ \|_{\mathscr{M}}$—the Minkowski functional of the set $\mathscr{M}$ which is a norm in $\mathscr{L}^*$. Hence we can equip the space $(\mathscr{L}^*)^* = \mathscr{L}$ with a norm $\|A\|_\varepsilon = \max_{\gamma \in Z} |\gamma(A)|$. It is easy to observe that $\|\gamma\|_{\mathscr{M}} = \sup_{\|A\|_\varepsilon \leqslant 1} |\gamma(A)|$ and, consequently, $\| \ \|_{\mathscr{M}}$ is the dual norm for $\| \ \|_\varepsilon$ in $\mathscr{L}^*$. Now we will show that for every $A \in \mathscr{L}$, $| \, \|A\| - \|A\|_\varepsilon| \leqslant \varepsilon \|A\|$. Of course, we can assume $A \neq 0$. Then

$$\begin{aligned} |1 - \|(A/\|A\|)\|_\varepsilon| &= | \, \|C_i\| - \|(A/\|A\|)\|_\varepsilon| \\ &\leqslant | \, \|C_i\| - \|C_i\|_\varepsilon| + | \, \|C_i\|_\varepsilon - \|(A/\|A\|)\|_\varepsilon|, \end{aligned}$$

where $C_i \in S_{\mathscr{L}}$ is chosen so that $\|(A/\|A\|) - C_i\| \leqslant \varepsilon/3$. Hence

$$\begin{aligned} |1 - \|(A/\|A\|)\|_\varepsilon| &\leqslant \|C_i - (A/\|A\|)\|_\varepsilon + | \, \|C_i\| - \|C_i\|_\varepsilon| \\ &\leqslant \|C_i - (A/\|A\|)\| + \varepsilon/3 \leqslant (2/3)\,\varepsilon, \end{aligned}$$

since by (2.3)

$$\|C_i\|_\varepsilon \leqslant \|C_i\| \leqslant \varphi_i(C_i w_i) + \varepsilon/3 \leqslant \|C_i\|_\varepsilon + \varepsilon/3$$

(by (2.4), $\varphi_i \otimes w_i \in Z_1 \subset Z$). Consequently,

$$| \, \|(A/\|A\|)\| - \|(A/\|A\|)\|_\varepsilon| \leqslant 2\varepsilon/3$$

and

$$| \, \|A\| - \|A\|_\varepsilon| < \varepsilon \|A\|.$$

Hence we get immediately

$$(1 - \varepsilon)\, \|A\| \leqslant \|A\|_\varepsilon \leqslant (1 + \varepsilon)\, \|A\|.$$

From this, it is easy to deduce that

$$|\text{dist}(L, v) - \text{dist}_\varepsilon(L, \mathscr{V})| \leqslant \varepsilon \, \|L\|.$$

(dist_ε denotes the distance of L from $\mathscr{V}$ with respect to the $\| \, \|_\varepsilon$.) Now let $V_0 \in \mathscr{P}_{\mathscr{V}}(L)$ (see 1.1) and let $V_\varepsilon \in \mathscr{P}^\varepsilon_{\mathscr{V}}(L)$ (the set of best approximants with respect to the $\| \, \|_\varepsilon$). By Theorem 1.3, $0 \in \text{conv}\, E_\varepsilon(L - V_\varepsilon)|_{\mathscr{V}}$ (see 1.1). It is evident by the definition of $\| \, \|_\varepsilon$ that $E_\varepsilon(L - V_\varepsilon) \subset \bigcup_{\alpha \in K, |\alpha| = 1} \alpha Z$. Hence $E_\varepsilon(L - V_\varepsilon) = \{\varphi_1 \otimes w_1|_{\mathscr{L}}, ..., \varphi_l \otimes w_l|_{\mathscr{L}}\}$, where $\varphi_i \in \text{ext}\, S_{V^*}$ and $w_i \in S_W$ for $i = 1, ..., l$. Note that

$$\begin{aligned}
\varepsilon \, \|L\| &\geqslant |\text{dist}(L, \mathscr{V}) - \text{dist}_\varepsilon(L, \mathscr{V})| \\
&= | \, \|L - V_0\| - \|L - V_\varepsilon\|_\varepsilon \, | \\
&= \left| \, \|L - V_0\| - \sum_{i=1}^{l} \lambda_i(\varphi_i \otimes w_i)(L - V_\varepsilon) \right| \\
&= \left| \, \|L - V_0\| - \sum_{i=1}^{l} \lambda_i(\varphi_i \otimes w_i)(L - V_0) \right|,
\end{aligned}$$

where $\lambda_i > 0$, $\sum_{i=1}^{l} \lambda_i = 1$, and $\sum_{i=1}^{l} \lambda_i(\varphi_i \otimes w_i)|_{\mathscr{V}} = 0$. This proves the first part of the theorem (if $\|L\| \neq 1$ we can start from $\varepsilon/\|L\|$).
Now suppose, on the contrary, that $V_0 \notin \mathscr{P}_{\mathscr{V}}(L)$ and condition (2.2) holds. Put $\varepsilon = (\|L - V_0\| - \text{dist}(L, \mathscr{V}))/2$ and let $V_1 \in \mathscr{P}_{\mathscr{V}}(L)$. Then

$$\varepsilon + \|L - V_1\| < \|L - V_0\| \leqslant \left| \sum_{i=1}^{l} \lambda_i(\varphi_i \otimes w_i)(L - V_0) \right| + \varepsilon,$$

which by (2.1) gives

$$\|L - V_1\| < \left| \sum_{i=1}^{l} \lambda_i(\varphi_i \otimes w_i)(L - V_1) \right|,$$

a contradiction.

Remark 2.2. In Theorem 2.1 the set $\text{ext}\, S_{V^*}$ can be replaced by any norming set $C \subset S_{V^*}$ and S_W by any norming set $D \subset S_{W^{**}}$. (A set $F \subset S_{V^*}$ is called a norming set iff $\|v\| = \sup_{f \in F} |f(v)|$ for every $v \in V$.)

Applying Theorem 2.1 we may prove a necessary condition for $\mathscr{V}$ to be a non-Chebyshev subspace. The method of the proof is similar to that of [8].

THEOREM 2.3. *Assume $\mathscr{V} \subset \mathscr{L}(W, V)$ is a non-Chebyshev finite-dimensional subspace (we consider the real case). Then there exists $D \in \mathscr{V}$, $\|D\| = 1$*

such that for every $\varepsilon > 0$ *there exists* $f_1, ..., f_m \in \operatorname{ext} S_{V^*}$ *and* $w_1, ..., w_m \in W$, $\sum_{i=1}^m \|w_i\| = 1$ *such that*

(a) $G = \sum_{i=1}^m (f_i \otimes w_i)|_{\mathscr{V}} = 0$;

(b) *if* $F \in \mathscr{L}^*(W, V)$ *and* $\|G \pm F\| \leqslant 1$ *then* $|F(D)| < \varepsilon$.

(c) $\sum_{i=1}^m |(f_i \otimes w_i)(D)| < \varepsilon$.

Proof. Since $\mathscr{V}$ is a non-Chebyshev subspace, there exists $L \in \mathscr{L}(W, V)$ such that $0, \pm D \in \mathscr{P}_{\mathscr{V}}(L)$, $\|D\| = 1$. This will be the required D. Now fix $\varepsilon > 0$. Applying Theorem 2.1, we can find $f_1, ..., f_m \in \operatorname{ext} S_{V^*}$, $u_1, ..., u_m \in S_W$, $\lambda_1, ..., \lambda_m \geqslant 0$, and $\sum_{i=1}^m \lambda_i = 1$ such that:

$$\sum_{i=1}^m \lambda_i (f_i \otimes u_i)|_{\mathscr{V}} = 0 \tag{2.5}$$

and

$$\left| \sum_{i=1}^m \lambda_i (f_i \otimes u_i)(L) - \|L\| \right| < \varepsilon/2. \tag{2.6}$$

Put for $i = 1, ..., m$, $w_i = \lambda_i u_i$. Now we check that $f_1, ..., f_m$ and $w_1, ..., w_m$ satisfy (a), (b), (c). Note that condition (a) is guaranteed by (2.5). To prove (b), fix $F \in \mathscr{L}^*(W, V)$, $\|F \pm G\| \leqslant 1$. Hence $(F \pm G)(L) \leqslant \|L\|$. Since $G|_{\mathscr{V}} = 0$, $G(L) \pm F(L - D) \leqslant \|L - D\| = \|L\|$. By (2.6), $|F(L)| < \varepsilon/2$ and $|F(L - D)| < \varepsilon/2$. Hence $|F(D)| < \varepsilon$.

To show (c), put

$$\begin{aligned} P &= \{i : (f_i \otimes w_i)(D) \geqslant 0\}, \\ P_1 &= \{i : (f_i \otimes w_i)(D) > 0\}, \\ U &= \{i : (f_i \otimes w_i)(D) < 0\}. \end{aligned} \tag{2.7}$$

If U (P_1 resp.) is empty, then by (2.5) P_1 (U resp.) is empty and (c) holds true. So assume that U and P_1 are nonempty. Hence, by (2.5),

$$\sum_{i \in P} |(f_i \otimes w_i)(D)| = \sum_{i \in U} |(f_i \otimes w_i)(D)|.$$

Now suppose that (c) does not hold. Then

$$\sum_{i \in P} |(f_i \otimes w_i)(D)| \geqslant \varepsilon/2$$

and

$$\sum_{i \in U} |(f_i \otimes w_i)(D)| \geqslant \varepsilon/2. \tag{2.8}$$

Put $\gamma_P = \sum_{i \in P} \|w_i\|$ and $\gamma_U = \sum_{i \in U} \|w_i\|$. By (2.7), $\gamma_P > 0$, $\gamma_U > 0$, and $\gamma_U + \gamma_P = 1$. Set

$$S_1 = \sum_{i \in P} (f_i \otimes w_i)(L), \qquad S_2 = \sum_{i \in U} (f_i \otimes w_i)(L).$$

By (2.6) $S_1 + S_2 > \|L\| - \varepsilon/2$. Thus either $S_1 > \gamma_P(\|L\| - \varepsilon/2)$ or $S_2 > \gamma_U(\|L\| - \varepsilon/2)$. Suppose that $S_1 > \gamma_P(\|L\| - \varepsilon/2)$. Then by (2.7) and (2.8)

$$\sum_{i \in P} (f_i \otimes w_i)(L + D) > \gamma_P \|L\| = \gamma_P \|L + D\|,$$

since $0 < \gamma_P < 1$. But for each $i \in P$,

$$(f_i \otimes w_i)(L + D) \leqslant \|w_i\| \; \|L + D\|. \tag{2.9}$$

By summing both sides of (2.9) we get a contradiction.

If $S_2 > \gamma_U(\|L\| - \varepsilon/2)$ then a similar argument using U and $L - D$ provides a contradiction. The proof of Theorem 2.3 is complete.

Now we consider the case of strong unicity.

THEOREM 2.4. *Let X be a normed real space and let $V \subset X$ be an n-dimensional subspace with a basis $v_1, ..., v_n$. Let $S \subset S_{X^*}$ be a norming set. Assume furthermore that there is $\delta > 0$ such that for every set $f_1, ..., f_n$ of linearly independent functionals from S*

$$|\det[f_i(v_j)]_{i,j=1,...,n}| > \delta > 0. \tag{2.10}$$

(det A denotes the determinant of a matrix A.) Then each $x \in X$ has a strongly unique best approximation in V.

Proof. Fix $x \in X \backslash V$ and consider $Z = [x] \oplus V$. Since Z as a finitely dimensional subspace is separable, we can assume that S is countable. By the totality of S over Z, we can choose $k_0 \in N$ such that $\{\phi_1, ..., \phi_{k_0}\}$ is total over Z. Hence for each $k \geqslant k_0$ we can equip Z with a norm

$$\|z\|_k = \max_{i=1,...,k} |s_i(z)| \qquad (S = \{s_1, s_2, ...\}).$$

By (2.10) V with $\| \; \|_k$ is an interpolating subspace of Z. Hence for $k \geqslant k_0$ there exists $v_k \in V$ which is a SUBA (see 1.3) for x with respect to the $\| \; \|_k$. By Theorem 1.3, $0 \in \operatorname{int} \operatorname{conv} E_k(x - v_k)|_V$ (see 1.1). (We consider the set $E_k(x - v_k)$ with respect to the $\| \; \|_k$.) By Carathéodory's theorem $0 = \sum_{i=1}^{n+1} \lambda_i^k f_i^k|_V$, where $f_1^k, ..., f_{n+1}^k \in E_k(x - v_k)$, $\lambda_i^k > 0$, and $\sum_{i=1}^{n+1} \lambda_i^k = 1$. Passing to a subsequence if necessary, we can assume $v_k \to v_0$, $\lambda_i^k \to \lambda_i$, and $f_i^k \to f_i \in S_{Z^*}$.

It is evident that $f_i(x-v_0)=\|x-v_0\|$ and $\sum_{i=1}^{n+1} \lambda_i f_i|_V=0$. Now we show that $\lambda_i>0$ for $i=1, ..., n+1$. Note that $\lambda_{i_0}>0$ for some $i_0 \in \{1, ..., n+1\}$, since $\sum_{i=1}^{n+1} \lambda_i=1$, $\lambda_i \geqslant 0$ for $i=1, ..., n+1$. We can assume $i_0=n+1$. By the Cramer rule,

$$\lambda_i^k=\lambda_{n+1}^k \cdot \Delta_i^k / \Delta_{n+1}^k \qquad \text{for} \quad i=1, ..., n, \tag{2.11}$$

where

$$\Delta_i^k=(-1)^{i+1} \det[f_l^k(v_j)]_{j=1, ..., n,\ l=1, ..., n+1,\ l \neq i}.$$

Hence, by (2.11), $1/|\lambda_i^k| \leqslant M/\delta \cdot 2/|\lambda_{n+1}|$ for k sufficiently large and $M>0$ independent of k. Consequently, $\lambda_i=\lim_{k \to \infty} \lambda_i^k>0$. Now take $w \in V \backslash\{0\}$. Since the set $\{f_1|_V, ..., f_{n+1}|_V\}$ is total over V, $f_{i_0}(w)<0$ for some $i_0 \in \{1, ..., n+1\}$. From this we derive that $f(w)<0$ for some $f \in \operatorname{ext}\{g \in S_{Z^*}: g(x-v_0)=\|x-v_0\|\}$. An easy calculation shows that $f \in E(x-v_0)$ (see 1.1). Note that a function $G: S_V \ni w \to \inf\{g(w): g \in E(x-v_0)\}$ is upper semicontinuous and, by the above reasoning, $G(w)<0$ for every $w \in S_V$. By the compactness of S_V we get $\sup\{G(w): w \in S_V\}=-r<0$. Now fix $v \in V \backslash\{0\}$ and take $f \in E(x-v_0)$ with $f(v/\|v\|)<G(v/\|v\|)+r/2$. Hence $f(v/\|v\|)<-r/2$ and consequently $f(v)<-r/2 \cdot\|v\|$.

By Theorem 1.2, v_0 is a SUBA for x in V, which completes the proof of the theorem.

Remark 2.5. By ([7, Theorem 3.3]) the term δ in (2.10) is essential. Here

$$S=\{e_i \otimes x: x \in \operatorname{ext} S_{l^\infty}, e_i \in \operatorname{ext} S_{l_1}\}.$$

EXAMPLE 2.6. Assume $W=V=c_0$. Let $A \in \mathscr{L}(W, V)$ be so chosen that for every $i \in N$, $x \in \operatorname{ext} S_{l^\infty}$,

$$|(Ax)_i|>\delta>0.$$

Then, by Theorem 2.4, each $L \in \mathscr{L}(W, V)$ has a strongly unique best approximation in [A]. (The set S is the same as in Remark 2.5.)

EXAMPLE 2.7. Assume $W=l_1$, $V=c_0$. Let $A \in \mathscr{L}(W, V)$ be represented as an infinite matrix $[A(i, j)]_{i, j=1,2, ...}$. If there exists $\delta>0$ such that for every $i, j \in N$ $|A(i, j)|>\delta>0$, then each $L \in \mathscr{L}(W, V)$ possesses a strongly unique best approximation in $[A]$. Here

$$S=\{e_i \otimes e_j: i, j=1,2, ..., e_i, e_j \in \operatorname{ext} S_{l_1}\}.$$

3. STRONG UNICITY IN $\mathscr{K}(c_0)$

We start with the following.

THEOREM 3.1. *Let $\mathscr{V} \subset \mathscr{K}(c_0)$ be a finite dimensional Chebyshev subspace. (The symbol $\mathscr{K}(c_0)$ denotes the space of all compact operators from c_0 into c_0; we consider the real case). Then each $L \in \mathscr{K}(c_0)$ has a strongly unique best approximation in $\mathscr{V}$.*

Proof. Assume that there exists $L_0 \in \mathscr{K}(c_0) \backslash \mathscr{V}$ such that $V_0 \in \mathscr{P}_{\mathscr{V}}(L_0)$ (see (1.2)) is not a SUBA for L_0 in $\mathscr{V}$. Put

$$I = \{i \in N: \|e_i \circ (L_0 - V_0)\| = \|L_0 - V_0\|\}. \tag{3.1}$$

(We denote $e_i(x) = x_i$ for $x \in c_0$.) By [9],

$$\text{ext } S_{\mathscr{K}^*(c_0)} = \text{ext } S_{l^1} \otimes \text{ext } S_{l^\infty}. \tag{3.2}$$

Hence

$$\|L_0 - V_0\| = (e_i \otimes x^i)(L_0 - V_0)$$

for all $e_i \otimes x^i \in E(L_0 - V_0)$ (see 1.1). Consequently, the set I is nonempty. For each $i \in I$ define

$$Z_i = \{x \in \text{ext } S_{l^\infty} : (e_i \otimes x)(L_0 - V_0) = \|L_0 - V_0\|\}. \tag{3.3}$$

Since $V_0 \in \mathscr{P}_{\mathscr{V}}(L_0)$, by Theorem 1.1, for every $V \in \mathscr{V}$ there exists $i \in I$ and $x^i \in Z_i$ such that

$$(e_i \otimes x^i)(V) \leqslant 0. \tag{3.4}$$

Since V_0 is not a SUBA for L_0 and $\mathscr{V}$ is finite dimensional, by Theorem 1.2, there exists $V_1 \in S_{\mathscr{V}}$ such that for every $i \in I$ and $x \in Z_i$

$$(e_i \otimes x)(V_1) \geqslant 0. \tag{3.5}$$

Now assume that we have constructed $L \in \mathscr{K}(c_0)$ such that

$$\|L - \alpha V_1\| \leqslant \|L\| \tag{3.6}$$

for $\alpha \in [0, \alpha_0)$ and

$$(e_i \otimes x)(L) = \|L\| \tag{3.7}$$

for every $i \in I$ and $x \in Z_i$. By Theorem 1.1, (3.4), and (3.6), $\alpha V_1 \in \mathscr{P}_{\mathscr{V}}(L)$ for every $\alpha \in [0, \alpha_0)$, which contradicts the fact that $\mathscr{V}$ is a Chebyshev subspace. So to finish the proof, it is necessary to construct an $L \in \mathscr{K}(c_0)$

satisfying (3.6) and (3.7). To do this, fix $i \in I$ and $x = (x_1, x_2, \ldots) \in Z_i$. If $\sum_{k=1}^{\infty} |V_1(i, k)| = 0$ (V_1 is represented by a matrix $[V_1(i, k)]_{i, k=1, 2, \ldots}$) then define

$$L_i = (L(i, k))_{k=1, 2, \ldots}, \tag{3.8}$$

where

$$L(i, k) = L_0(i, k) - V_0(i, k).$$

(Here $[L_0(i, k)]_{i, k=1, 2, \ldots}$ denote the matrix corresponding to L_0 and $[V_0(i, k)]_{i, k=1, 2, \ldots}$ the matrix corresponding to V_0).

If $\sum_{k=1}^{\infty} |V_1(i, k)| > 0$, then put

$$U_i = \{k \in N \colon L_0(i, k) - V_0(i, k) = 0\}. \tag{3.9}$$

Since $\|e_i \circ (L_0 - V_0)\| = \operatorname{dist}(L_0, \mathscr{V}) > 0$, $U_i \neq N$. Put

$$F_i = \{k \in N \setminus U_i \colon x_k = \operatorname{sgn} V_1(i, k)\}, \tag{3.10}$$

$$E_i = N \setminus (U_i \cup F_i). \tag{3.11}$$

Take $y = (y_1, y_2, \ldots,) \in \operatorname{ext} S_{l^\infty}$ given by

$$y_k = \begin{cases} x_k & \text{for} \quad k \in F_i \cup E_i \\ -\operatorname{sgn} V_1(i, k) & \text{for} \quad k \in U_i. \end{cases} \tag{3.12}$$

By (3.9) and (3.12), $(e_i \otimes y)(L_0 - V_0) = \|L_0 - V_0\|$. According to (3.5),

$$(e_i \otimes y)(V_1) = \sum_{k \in F_i} |V_1(i, k)| - \sum_{k \in (U_i \cup E_i)} |V_1(i, k)| \geqslant 0. \tag{3.13}$$

From this we derive $F_i \neq \varnothing$, since $\sum_{k=1}^{\infty} |V_1(i, k)| > 0$. Define for $k \in N$,

$$L(i, k) = \begin{cases} V_1(i, k) & \text{for} \quad k \in F_i \\ 0 & \text{for} \quad k \in N \setminus F_i \end{cases} \tag{3.14}$$

and set $L_i = (L(i, 1), L(i, 2), \ldots)$. We show that for $\alpha \in [0, 1)$, $\beta \geqslant 1$,

$$\sum_{k=1}^{\infty} |\beta L(i, k) - \alpha V_1(i, k)| \leqslant \beta \cdot \sum_{k=1}^{\infty} |L(i, k)|. \tag{3.15}$$

To do this, take any $z \in \operatorname{ext} S_{l_\infty}$. If $z_k = x_k$ for every $k \in F_i$ then

$$(e_i \otimes z)(V_1) = \sum_{k=1}^{\infty} V_1(i,k)\, z_k = \sum_{k \in F_i} |V_1(i,k)| + \sum_{k \in E_i \cup U_i} V_1(i,k)\, z_k \geqslant 0$$

by (3.13). Hence

$$\begin{aligned}\sum_{k=1}^{\infty} (\beta L(i,k) - \alpha V_1(i,k))\, z_k &= \sum_{k=1}^{\infty} \beta L(i,k)\, z_k - \alpha \cdot \sum_{k=1}^{\infty} V_1(i,k)\, z_k \\ &= \beta \cdot \sum_{k \in F_i} |L(i,k)| - \alpha \cdot \sum_{k=1}^{\infty} V_1(i,k)\, z_k \\ &\leqslant \beta \cdot \sum_{k=1}^{\infty} |L(i,k)|.\end{aligned}$$

If $z_k = -x_k$ for some $k \in F_i$, then the set $F_i^1 = \{k \in F_i \colon x_k = -z_k\}$ is non-empty. Compute

$$\begin{aligned}&\sum_{k=1}^{\infty} (\beta L(i,k) - \alpha V_1(i,k))\, z_k \\ &= \sum_{k \in F_i^1} (\beta L(i,k) - \alpha V_1(i,k))\, z_k + \sum_{k \in (F_i \setminus F_i^1) \cup E_i \cup U_i} (\beta L(i,k) - \alpha V_1(i,k))\, z_k \\ &= \sum_{k \in F_i^1} (\alpha - \beta)\, |V_1(i,k)| + \sum_{k \in (F_i \setminus F_i^1) \cup E_i \cup U_i} (\beta L(i,k) - \alpha V_1(i,k))\, z_k \\ &\leqslant \sum_{k \in F_i^1} (\beta - \alpha)\, |V_1(i,k)| + \sum_{k \in F_i \setminus F_i^1} (\beta - \alpha)\, |V_1(i,k)| + \sum_{k \in E_i \cup U_i} -\alpha V_1(i,k)\, z_k \\ &= \sum_{k \in F_i} \beta\, |V_1(i,k)| - \alpha \cdot \left(\sum_{k \in F_i} |V_1(i,k)| + \sum_{k \in E_i \cup U_i} V_1(i,k)\, z_k \right) \\ &\leqslant \beta \cdot \sum_{k \in F_i} |(V_1(i,k)| \\ &= \beta \cdot \sum_{k=1}^{\infty} |L(i,k)|\end{aligned}$$

(see 3.13).

Now if $i \notin I$ then we define $L_i = (L(i,1), L(i,2) \ldots,)$ by

$$L(i,k) = L_0(i,k) - V_0(i,k) \qquad \text{for} \quad k = 1, 2, \ldots. \tag{3.16}$$

Finally observe that by the Schur theorem (see [4, p. 864]) for $i \geqslant i_0$

$$\|e_i \circ (L_0 - V_0)\| \leqslant \operatorname{dist}(L_0, \mathscr{V})/2.$$

Hence the set I is finite and

$$M = \sup_{i \in N \setminus I} \|e_i \circ (L_0 - V_0)\| < \|L_0 - V_0\|. \tag{3.17}$$

Following (3.15) for $i \in I$ we can modify, if necessary, the rows L_i defined by (3.8) and (3.14), multiplying them by constants $\beta_i \geqslant 1$ such that

$$\|L_i - \alpha V_1(i, \cdot)\|_1 < \|L_i\|_1 = a > \|L_0 - V_0\|$$

for $\alpha \in [0, 1)$. Now choose $\alpha_0 \in (0, 1)$ such that $M + \alpha_0 < \|L_0 - V_0\|$. By (3.17), for $\alpha \in [0, \alpha_0)$ and $i \in N \setminus I$,

$$\|L_i - \alpha V_1(i,)\|_1 < \|L_0 - V_0\|.$$

Hence, by following (3.8), (3.14), and (3.15), the operator L defined by (3.8), (3.14), and (3.16) satisfies (3.6) for $\alpha \in [0, \alpha_0)$ and (3.7) for all $i \in I$ and $x \in Z_i$. The proof of Theorem 3.1 is complete.

Note that the unicity of best approximation for given $L \in \mathscr{K}(c_0)$ in $\mathscr{V}$ does not force the strong unicity because of

EXAMPLE 3.2. Let $L = [L(i, k)]_{i, k = 1, 2, \ldots}$ and $V = [V(i, k)]_{i, k = 1, 2, \ldots}$ be defined by

$$L(i, k) = \begin{cases} 0 & \text{if } i \neq 1 \\ 1/k^3 & \text{if } i = 1 \end{cases}$$

$$V(i, k) = \begin{cases} 0 & \text{if } i \neq 1 \\ (-1)^k/k^2 & \text{for } i = 1, \quad k > 1 \\ -\sum_{l=2}^{\infty} (-1)^l/l^2 & \text{for } i = 1, \quad k = 1 \end{cases}$$

Let $\mathscr{V} = [V]$. We show that 0 is the unique best approximation for L in $\mathscr{V}$. Take $\alpha \in R \setminus \{0\}$. If $\alpha > 0$, choose an even number k_0 such that $\alpha/k_0^2 > 1/k_0^3$. Let $z = (z_1, z_2, \ldots) \in \operatorname{ext} S_{l^\infty}$ be given by

$$z_k = \begin{cases} 1 & \text{if } k \neq k_0 \\ -1 & \text{if } k = k_0 \end{cases}.$$

Then

$$\|L-\alpha V\| \geqslant (e_1 \otimes z)(L-V)$$
$$= \sum_{l=1}^{\infty} z_l(L(1,l)-\alpha V(1,l))$$
$$= \sum_{l=1}^{\infty} (L(1,l)-V(1,l)) + 2(\alpha/k_0^2 - 1/k_0^3)$$
$$> \sum_{l=1}^{\infty} L(1,l) = \|L-0\|,$$

since $\sum_{l=1}^{\infty} V(1,l)=0$. If $\alpha<0$, choose k_0 odd such that $-\alpha/k_0^2 > 1/k_0^3$.

Reasoning as above we get $\|L-\alpha V\| > \|L\|$. Hence $0 \in \mathscr{P}_{\mathscr{V}}(L)$ is the unique best approximation. However, $E(L-0)=\{e_1 \otimes (1,1,...)\}$ (see (3.2) and (1.1)). Since $e_1 \otimes (1,1,...)(V)=0$, by Theorem 1.2 0 is not a SUBA for L in $\mathscr{V}$.

Remark 3.3. If we replace c_0 by l_∞^m, then by [3, Theorem 2.2(a)] or [9] the set ext $S_{\mathscr{K}^*(l_\infty^m)}$ is finite. By [6], if $\mathscr{V}$ is a subspace of $\mathscr{K}(l_\infty^m)$ then $L \in \mathscr{K}(l_\infty^m)$ has a unique best approximation in $\mathscr{V}$ if and only if L has a strongly unique best approximation in $\mathscr{V}$.

COROLLARY 3.4. *If $V \in S_{\mathscr{K}(c_0)}$ then $\mathscr{V}=[V]$ is a Chebyshev subspace if and only if for every $i \in N$ and $x \in \text{ext}\, S_{l^1}$,*

$$(e_i \otimes x)(V) \neq 0. \tag{3.18}$$

Comparing Corollary 3.4 with Theorem 3.3 of [7] we get

PROPOSITION 3.5. *There exists*

$$\varphi \in \text{ext}\, S_{\mathscr{L}^*(c_0)} \backslash \{e_i \otimes x: i=1,2,...,x \in \text{ext}\, S_{l^1}\}.$$

Proof. If $\text{ext}\, S_{\mathscr{L}^*}(c_0) \subset \{(e_i \otimes x): i=1,2,...,x \in \text{ext}\, S_{l^1}\}$ then by Theorems 1.1 and 1.2 each V satisfying (3.18) defines a Chebyshev subspace in $\mathscr{L}(c_0)$ which contradicts Theorem 3.3 of [7].

Proposition 3.5 shows that Theorem 2.2(a) of [3] cannot be generalized from the case compact operators to the case of linear operators.

At the end of this section we present an example of a two-dimensional Chebyshev subspace in $\mathscr{K}(c_0)$. The reasoning presented here is similar to that of [1]. First we recall, after [1],

LEMMA 3.6. *Let $M>1$ be given. Assume $f(r)=\sum_{n=0}^{\infty} a_n r^n$ is a power series whose coefficients are not all 0. Assume that if $a_n \neq 0$ then*

$$1 \leqslant |a_n| \leqslant M.$$

Then for every $r \in (0, 1/(M+1))$, $f(r) \neq 0$.

Proof. Let N denote the smallest index n such that $a_n \neq 0$. Then

$$\begin{aligned} |f(r)| = \left| \sum_{n=N}^{\infty} a_n r^n \right| &\geqslant |a_N \cdot r^N| - \sum_{n=N+1}^{\infty} |a_n| \, |r|^n \\ &\geqslant |r|^N - M \, |r|^{N+1}/(1-|r|) \\ &= |r|^N/(1-|r|)(1-|r|(1+M)) > 0. \end{aligned}$$

EXAMPLE 3.7. Let $c \in (0, 1)$, $r \in (0, 1/4)$. Define

$$V_1(i, k) = c^i \cdot r^{2^{2k+1}} \quad \text{for} \quad i, k = 1, 2, ..., \tag{3.19}$$

$$V_2(i, k) = (c/2)^i \cdot r^{2^{2k+2}} \quad \text{for} \quad i, k = 1, 2, ... \tag{3.20}$$

We show that V_1, V_2 defined by (3.19) and (3.20) form a two-dimensional interpolating (hence Chebyshev) subspace in $\mathscr{K}(c_0)$. To do this, take $\varphi_1 = e_{i_1} \otimes x_1$, $\varphi_2 = e_{i_2} \otimes x_2$ to be two linearly independent functionals from $\operatorname{ext} S_{\mathscr{K}^*(c_0)}$. We prove that $\det[\varphi_i(V_j)]_{i,j=1,2} \neq 0$. Let $x_j = (\sigma_{1j}, \sigma_{2j}, ...)$ for $j = 1, 2$ ($\sigma_{ij} = \pm 1$). Note that

$$\begin{aligned} \det[\varphi_i(V_j)]_{i,j=1,2} &= \det \begin{bmatrix} \sum_{j=1}^{\infty} \sigma_{1j} V_1(i_1, j), \ \sum_{j=1}^{\infty} \sigma_{1j} V_2(i_1, j) \\ \sum_{j=1}^{\infty} \sigma_{2j} V_1(i_2, j), \ \sum_{j=1}^{\infty} \sigma_{2j} V_2(i_2, j) \end{bmatrix} \\ &= \sum_{j_1, j_2 = 1}^{\infty} \det \begin{bmatrix} \sigma_{1j_1} V_1(i_1, j_1), \sigma_{1j_2} V_2(i_1, j_2) \\ \sigma_{2j_1} V_1(i_2, j_1), \sigma_{2j_2} V_2(i_2, j_2) \end{bmatrix} \\ &= \sum_{j_1, j_2 = 1}^{\infty} \det \begin{bmatrix} \sigma_{1j_1} c^{i_1} r^{2^{2j_1+1}}, \sigma_{1j_2} (c/2)^{i_1} r^{2^{2j_2+2}} \\ \sigma_{2j_1} c^{i_1} r^{2^{2j_1+1}}, \sigma_{2j_2} (c/2)^{i_2} r^{2^{2j_2+2}} \end{bmatrix} \\ &= \sum_{j_1, j_2 = 1}^{\infty} r^{2^{2j_1+1} + 2^{2j_2+2}} \cdot \det \begin{bmatrix} \sigma_{1j_1} c^{i_1}, \sigma_{1j_2} (c/2)^{i_1} \\ \sigma_{2j_1} c^{i_2}, \sigma_{2j_2} (c/2)^{i_2} \end{bmatrix}. \end{aligned}$$

If $2^{2j_1+1} + 2^{2j_2+2} = 2^{2k_1+1} + 2^{2k_2+2}$, because of the unique binary expression of each integer we get $j_1 = k_1$ and $j_2 = k_2$. In particular, then, distinct pairs (j_1, j_2) give distinct powers of r. Hence the above determinant can be regarded as a power series with coefficients.

$$A_{j_1, j_2} = \det \begin{bmatrix} \sigma_{1j_1} c^{i_1}, \sigma_{1j_2} (c/2)^{i_1} \\ \sigma_{2j_1} c^{i_2}, \sigma_{2j_2} (c/2)^{i_2} \end{bmatrix}.$$

If $i_1 = i_2$ then

$$\det[\varphi_i(V_j)]_{i,j=1,2} = (c^2/2)^{i_1} \cdot \sum_{j_1, j_2 = 1}^{\infty} r^{2^{2j_1+1} + 2^{2j_2+2}} B_{j_1, j_2},$$

where

$$B_{j_1, j_2} = \det \begin{bmatrix} \sigma_{1, j_1}, \sigma_{1, j_2} \\ \sigma_{2, j_1}, \sigma_{2, j_2} \end{bmatrix}. \tag{3.21}$$

Since $e_{i_1} \otimes x_1$, $e_{i_2} \otimes x_2$ are linearly independent, not all B_{j_1, j_2} are equal to 0. Note that if $B_{j_1, j_2} \neq 0$ then $|B_{j_1, j_2}| = 2$. If $i_1 \neq i_2$ (we may assume $i_1 < i_2$) then

$$\det[\varphi_i(V_j)]_{i, j=1,2} = c^{i_1 + i_2}[(1/2)^{i_1} - (1/2)^{i_2}] \cdot \sum_{j_1, j_2 = 1}^{\infty} r^{2^{2j_1+1} + 2^{2j_2+2}} B_{j_1, j_2},$$

where

$$B_{j_1, j_2} = (1/[(1/2)^{i^1} - (1/2)^{i_2}]) \cdot \det \begin{bmatrix} \sigma_{1j_1}, \sigma_{1j_2}(1/2)^{i_1} \\ \sigma_{2j_1}, \sigma_{2j_2}(1/2)^{i_2} \end{bmatrix}. \tag{3.22}$$

It is clear that

$$\begin{aligned} 1 \leqslant |B_{j_1, j_2}| &\leqslant [(1/2)^{i_1} + (1/2)^{i_2}]/[(1/2)^{i_1} - (1/2)^{i_2}] \\ &= [1 + (1/2)^{i_2 - i_1}]/[1 - (1/2)^{i_2 - i_1}] \\ &\leqslant [1 + (1/2)]/[1 - (1/2)] = 3. \end{aligned}$$

Applying Lemma 3.6 to the series

$$\sum_{j_1, j_2 = 1}^{\infty} B_{j_1, j_2} r^{2^{2j_1+1} + 2^{2j_2+2}},$$

where B_{j_1, j_2} are defined by (3.21) or (3.22), we get $\det[\varphi_i(V_j)]_{i, j=1,2} \neq 0$ as required.

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